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ON MULTISET k-FAMILIES

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The generalized Macaulay theorem is extended slightly to a form convenient for dealing with k-families of subsets of a multiset. Several applications are given.

1. Introduction

A finite multiset M is a finite set consisting of k_i elements of type i, $i = 1, 2, \ldots, n$, where we may assume $k_1 \geqslant \cdots \geqslant k_n \geqslant 1$. For example, M might be a set of $k_1 + \cdots + k_n = K_n$ billiard balls, k_i of color i. A k-family is a collection of subsets of M, no $k + 1$ of which can be linearly ordered by set-wise inclusion. The "k" in "k-family" is completely unrelated to any subscripted "k" indicating multiplicity of elements.

The generalized Macaulay theorem is a versatile result useful for dealing with 1-families (also called antichains). It involves the *shadow* operator Γ (or Δ). The shadow ΓX of a collection X of subsets of M is the set of all subsets of M which can be obtained by removing a single element of M from a member of X. The *rank* of a subset of M is the number of elements of M it contains.

Now suppose one wishes to choose a fixed number m of subsets of M of a given rank so that the shadow of the selection is as small as possible. Which ones should be chosen? The generalized Macaulay theorem [1] specifies an ordering for the subsets of M under which the answer is, the first m subsets of the given rank.

The ith rank parameter of a collection of subsets of M is the number of subsets in the collection having rank i. In particular, the ith rank parameter of the entire collection of subsets of M is called the ith *Whitney number* [11, p. 24]. A characterization of the rank parameters of a 1-family of subsets of a multiset follows easily from the generalized Macaulay theorem [2]. Daykin [9, Theorem 12] has characterized the rank parameters of k-families of subsets of ordinary finite sets and Clements [5, Theorem 12] has verified that his result extends to multisets.

In this paper the generalized Macaulay theorem is extended slightly to a form that is useful in dealing with k-families (Theorem 1). As an application, we characterize the rank parameters of a k-family in terms of the rank parameters of k 1-families (Theorem 3) and as a corollary are able to verify a conjecture of

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Daykin [9, p. 94]. Theorem 2 is a reformulation of the Daykin–Clements characterization of the rank parameters of a k-family mentioned above.

2. An extension of the generalized Macaulay theorem

We identify the subset of $M(k_1, \ldots, k_n) = M(\boldsymbol{k})$ consisting of x_i elements of type i, $0 \leqslant x_i \leqslant k_i$, with the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ and use $S(\boldsymbol{k}) = S$ to denote the set of all such vectors. (S may also be regarded as the set of divisors of the integer $p_1^{k_1} \cdots p_n^{k_n}$, where $p_1, \ldots, p_n$ are distinct primes and $\boldsymbol{x}$ is identified with the divisor $p_1^{x_1} \cdots p_n^{x_n}$.) Evidently $|S| = (k_1 + 1) \cdots (k_n + 1)$, and when $k_1 = 1$ (and therefore all k_i are 1 in view of the assumption $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n \geqslant 1$) M is an ordinary n-element set.

We order the vectors of S by defining $\boldsymbol{x} < \boldsymbol{y}$ if $x_i < y_i$ for the largest integer i satisfying $x_i \neq y_i$ (reverse lexicographic order).

The *rank* $r(\boldsymbol{x})$ of $\boldsymbol{x}$ is $x_1 + \cdots + x_n$. It is useful to imagine the elements of S arrayed by writing them in increasing order from left to right, top to bottom with $k_1 + 1$ elements in a row and element $\boldsymbol{x}$ in column $r(\boldsymbol{x})$. For example, $S(2, 1, 1)$ is exhibited in Fig. 1.

For any subset H of S, H_i denotes the vectors of rank i of H. $F(m, H)$ and $L(m, H)$ denote respectively the first and last m elements of H. In particular, $CH_i = F(|H_i|, S_i)$ is the *compression* of H_i.

The *shadow* $\Gamma\boldsymbol{x}$ of an element $\boldsymbol{x}$ of S is

$$\Gamma\boldsymbol{x} = S \cap \{(x_1 - 1, x_2, \ldots, x_n),$$
$$(x_1, x_2 - 1, x_3, \ldots, x_n), \ldots, (x_1, x_2, \ldots, x_{n-1}, x_n - 1)\}$$

and the shadow ΓX of a subset X of S is $\bigcup_{\boldsymbol{x} \in X} \Gamma\boldsymbol{x}$. By the generalized Macaulay theorem, for any integer l, $0 \leqslant l \leqslant K_n$ and any subset H of S,

$$|\Gamma CH_l| \leqslant |\Gamma H_l|.$$

For example (see Fig. 1), in $S(2, 1, 1)$, $2 = |\Gamma\{200, 110\}| \leqslant |\Gamma\{110, 011\}| = 3$.

We abbreviate the Whitney number $|S_l(\boldsymbol{k})|$ to $\langle {n \atop l} \rangle \boldsymbol{k}$, or simply $\langle {n \atop l} \rangle$ when there is no danger of confusion, $l = 0, 1, 2, \ldots, K_n$. Since S_l is just the set of ordered n-part partitions $(x_1, \ldots, x_n)$ of l, with part j satisfying $0 \leqslant x_j \leqslant k_j$, $\langle {n \atop l} \rangle$ is the

Rank	0	1	2	3	4
	000	100	200		
		010	110	210	
		001	101	201	
			011	111	211

Fig. 1. $S(2, 1, 1)$.

coefficient of x^l in $\prod_{j=1}^{n}(1+x+x^2+\cdots+x^{k_j})$. Thus, in the case of ordinary sets, $\langle {n \atop l} \rangle$ is the usual binomial coefficient $\binom{n}{l}$.

Our abbreviation of the Whitney numbers $|S_l(k)|$ to $\langle {n \atop l} \rangle$ is felicitous in that facts about the binomial coefficients $\binom{n}{l}$ often generalize naturally to facts about the coefficients $\langle {n \atop l} \rangle$, which can be regarded as generalized binomial coefficients. For example, the familiar recurrence

$$\binom{n}{l} = \sum_{j=0}^{k_n} \binom{n-1}{l-j}, \quad \text{except if } n = l = 0,$$

(where $\binom{0}{0}$ is understood to be 1) generalizes to

$$\left\langle {n \atop l} \right\rangle = \sum_{j=0}^{k_n} \left\langle {n-1 \atop l-j} \right\rangle, \quad \text{except if } n = l = 0$$

(where $\langle {0 \atop 0} \rangle$ is understood to be 1). This follows immediately from

$$\prod_{j=1}^{n}(1+x+x^2+\cdots+x^{k_j}) = (1+x+x^2+\cdots+x^{k_n}) \prod_{j=1}^{n-1}(1+x+x^2+\cdots+x^{k_j}).$$

The above recurrence permits us to list the generalized binomial coefficients $\langle {i \atop l} \rangle$, $i = 0, 1, \ldots, n$, $l = 0, 1, \ldots, K_i = k_1 + k_2 + \cdots + k_i$ corresponding to $S(k)$ in a Pascal's triangle. The coefficients corresponding to $S(2, 1, 1)$ are given in Fig. 2. Comparison of Figs. 1 and 2 reveals that $\langle {n-i \atop l} \rangle$ is the number of vectors x of rank l in $S(k)$ with $x_n = x_{n-1} = \cdots = x_{n-i+1} = 0$.

In working with k-families of subsets of M, or more briefly, k-families in S, it can be helpful to consider k copies of S arranged vertically. For example, if we use kS to denote such an array, $3S(2, 1, 1)$ is exhibited in Fig. 3.

We denote the ith copy of S in kS by kS^i, and the vector x in kS^i by $(x; i)$, $i = 1, 2, \ldots, k$. The set kS is partially ordered by taking $(x; i) < (y; j)$ if and only if $i = j$ and $x < y$. Thus kS is not quite the same as the product of S with the k-element chain $\{1, 2, \ldots, k\}$ in which distinct $(x; i)$ and $(x; j)$ are related. The rank of $(x; i)$ is just the rank of x. We continue to use H_l for the elements of rank l of a subset H of kS and we write kS_l in place of $(kS)_l$. $CH_l = F(|H_l|, kS_l)$, the first (counting from the top) $|H_l|$ elements of kS_l.

	$\langle {} \atop 0 \rangle$	$\langle {} \atop 1 \rangle$	$\langle {} \atop 2 \rangle$	$\langle {} \atop 3 \rangle$	$\langle {} \atop 4 \rangle$
$\langle {0 \atop} \rangle$	1				
$\langle {1 \atop} \rangle$	1	1	1		
$\langle {2 \atop} \rangle$	1	2	2	1	
$\langle {3 \atop} \rangle$	1	3	4	3	1

Fig. 2. Generalized binomial coefficients corresponding to $S(2, 1, 1)$.

Rank	0	1	2	3	4
	000	100	200		
		010	110	210	
		001	101	201	
			011	111	211
	000	100	200		
		010	110	210	
		001	101	201	
			011	111	211
	000	100	200		
		010	110	210	
		001	101	201	
			011	111	211

Fig. 3. $3S(2, 1, 1)$.

The shadow operator Γ, which is defined on S, is extended to an operator $k\Gamma$ defined on kS by restricting Γ to copies: for $(x; i) \in kS^i$, $k\Gamma(x; i)$ is defined to be $\{(y; i) \mid y \in \Gamma x\}$. For a subset X of kS, $k\Gamma X$ is $\bigcup_{x \in X} k\Gamma x$. Since the "$k$" in "$k\Gamma$" is usually fixed, we will often suppress it.

The first several elements of a rank in kS are called an initial segment. Since the shadow of an initial segment in S is an initial segment [1; Lemma 3], the same is true in kS. Γ^j is defined inductively by $\Gamma^1 = \Gamma$ and $\Gamma^j = \Gamma\Gamma^{j-1}$, $j = 2, 3, \ldots$.

The obvious analogue of the generalized Macaulay theorem (Theorem 1 below) holds in kS: an m-element subset of kS_l with minimal shadow is the m-element initial segment. To prove it, we first formulate a lemma.

For $1 \leqslant l \leqslant K_n$, $0 \leqslant a < \langle {n \atop l} \rangle$, and $0 < b \leqslant \langle {n \atop l} \rangle - a$, $\Gamma F(a+b, S_l) \setminus \Gamma F(a, S_l)$ might be called the *penumbra* of the consecutive b elements $F(a+b, S_l) \setminus F(a, S_l)$ since it consists of the elements in the shadow of $F(a+b, S_l)$ which are not also in the shadow of $F(a, S_l)$. We now see that the first and last b elements of S_l are respectively the consecutive-b-element subsets of S_l with the largest and smallest penumbras.

Lemma. *If* $1 \leqslant l \leqslant K_n$, $0 \leqslant a < \langle {n \atop l} \rangle$ *and* $0 < b \leqslant \langle {n \atop l} \rangle - a$, *then*

$$|\Gamma F(b, S_l)| \geqslant |\Gamma F(a+b, S_l)| - |\Gamma F(a, S_l)|$$
$$\geqslant \left|\Gamma F\left(\left\langle {n \atop l} \right\rangle, S_l\right)\right| - \left|\Gamma F\left(\left\langle {n \atop l} \right\rangle - b, S_l\right)\right|.$$

These inequalities were proved by Clements [3], but his induction proof was unnecessarily tedious. Kleitman [8, p. 246] pointed out that the left inequality, which is referred to as the subadditivity of $|\Gamma F(\cdot, S_l)|$, is actually a corollary of the generalized Macaulay theorem. In similar fashion, the right inequality is also: Let $S' = S(k_1, \ldots, k_n, 1, 1)$ and let X, Y and Z denote respectively the subsets of S'_{l+1} obtained by following each element of S_{l+1} by 00, following each element of $F(a+b, S_l)$ with 10, and following each element of $F(\langle {n \atop l} \rangle - b, S_l)$ by 01. The result then follows from

$$\begin{aligned}|\Gamma X| + |\Gamma F(a+b, S_l)| + \left|\Gamma F\left(\left\langle {n \atop l} \right\rangle - b, S_l\right)\right| \\ &= |\Gamma(X \cup Y \cup Z)| \\ &\geqslant \left|\Gamma F\left(|X| + \left\langle {n \atop l} \right\rangle + a, S'_{l+1}\right)\right| \\ &= |\Gamma X| + \left|\Gamma F\left(\left\langle {n \atop l} \right\rangle, S_l\right)\right| + |\Gamma F(a, S_l)|.\end{aligned}$$

The inequality here follows from the generalized Macaulay theorem for S'_{l+1}; the first equality follows since the set of vectors in ΓY with final components 10 has the same size as $\Gamma F(a+b, S_l)$, while the vectors with final components 00 are already counted in $|\Gamma X|$, etc.

Theorem 1. *For any subset H of kS and any integer l, $0 \leqslant l \leqslant K_n$,*

$$|\Gamma C H_l| \leqslant |\Gamma H_l|.$$

Proof. We proceed by induction on k. For $k = 1$, our theorem is just the generalized Macaulay theorem [1], so we assume the theorem for $k = 1, 2, \ldots, t-1$ and consider the case $k = t$. Let m_1 and m_2 denote the number of elements of H_l in the first $t-1$ copies and the last copy of S respectively. If m_1 or m_2 is 0, the theorem follows immediately from the induction hypothesis, so we henceforth assume that both are >0.

We will say that subsets H and G of kS are *isomorphic* ($H \sim G$) if one can be obtained from the other by permuting the copies of S; more precisely $H \sim G$ if the sets $H \cap kS^i$, $i = 1, \ldots, k$, regarded as subsets of S, are the same as the sets $G \cap kS^i$, $i = 1, \ldots, k$ except for order. Note that if $H \sim G$, then $\Gamma H \sim \Gamma G$; in particular, $|\Gamma H| = |\Gamma G|$.

If

$$H'_l = F(m_1, tS_l \setminus tS^t_l) \cup F(m_2, tS^t_l),$$

then

$$|\Gamma H'_l| \leqslant |\Gamma H_l|$$

by the $k = t-1$ and $k = 1$ instances of the induction hypothesis.

Now write

$$m_1 = s\binom{n}{l} + r, \quad \text{where } s = \left\lfloor m_1 \Big/ \binom{n}{l} \right\rfloor.$$

If $r=0$, $CH_l \sim H_l'$ and $|\Gamma CH_l| = |\Gamma H_l'| \leqslant |\Gamma H_l|$, while if $0 < r < \binom{n}{l}$, then

$$H_l' \sim F\left(s\binom{n}{l}, kS_l\right) \cup F(r, kS_l^{s+1}) \cup F(m_2, kS_l^{s+2})$$

and

$$\begin{aligned}|\Gamma CH_l| &= \left|\Gamma F\left(s\binom{n}{l}, kS_l\right)\right| + |\Gamma F(r+m_2, kS_l^{s+1} \cup kS_l^{s+2})| \\ &\leqslant \left|\Gamma F\left(s\binom{n}{l}, kS_l\right)\right| + |\Gamma F(r, kS_l^{s+1})| + |\Gamma F(m_2, kS_l^{s+2})| \\ &= |\Gamma H_l'| \leqslant |\Gamma H_l|.\end{aligned}$$

The first inequality here follows from the $k=2$ instance of the induction hypothesis. Thus the induction is complete, unless $t=2$, in which case

$$H_l' = F(m_1, 2S_l^1) \cup F(m_2, 2S_l^2).$$

If $m_1 + m_2 \leqslant \binom{n}{l}$, we have by the subadditivity of Γ that

$$\begin{aligned}|\Gamma CH_l| = |\Gamma F(m_1+m_2, S_l)| &\leqslant |\Gamma F(m_1, S_l)| + |\Gamma F(m_2, S_l)| \\ &= |\Gamma H_l'| \leqslant |\Gamma H_l|.\end{aligned}$$

Finally, if $m_1 + m_2 > \binom{n}{l}$, note that the penumbra X of the last $\binom{n}{l} - m_1$ elements in $2S_l^1$ is isomorphic to the penumbra Y of the last $\binom{n}{l} - m_1$ sets in $2S_l^2$ which, by the lemma (with $a = m_1 + m_2 - \binom{n}{l}$ and $b = \binom{n}{l} - m_1$), does not exceed in size the penumbra Z of the $(\binom{n}{l} - m_1)$-element set

$$F\left(m_1 + m_2 - \binom{n}{l} + \binom{n}{l} - m_1, 2S_l^2\right) \Big\backslash F\left(m_1 + m_2 - \binom{n}{l}, 2S_l^2\right).$$

In notation this reads

$$\begin{aligned}&|\Gamma CH_l| - |\Gamma F(m_1, 2S_l^1)| - \left|\Gamma F\left(m_1 + m_2 - \binom{n}{l}, 2S_l^2\right)\right| \\ &\quad = |X| = |Y| \leqslant |Z| \\ &\qquad\qquad = |\Gamma F(m_2, 2S_l^2)| - \left|\Gamma F\left(m_1 + m_2 - \binom{n}{l}, 2S_l^2\right)\right|\end{aligned}$$

or

$$\begin{aligned}|\Gamma CH_l| &\leqslant |\Gamma F(m_1, 2S_l^1)| + |\Gamma F(m_2, 2S_l^2)| \\ &= |H_l'| \leqslant |\Gamma H_l|.\end{aligned}$$

This completes the proof of Theorem 1. □

For given l $0 \leq l \leq K_n$ and m, $1 \leq m \leq k\langle {n \atop l} \rangle$, the lower bound $|\Gamma CH_l|$ appearing in Theorem 1, where $m = |CH_l|$, can be calculated as follows: if $m = s\langle {n \atop l} \rangle + r$, where $s = \lfloor m/\langle {n \atop l} \rangle \rfloor$, then

$$|\Gamma CH_l| = s\left\langle {n \atop l-1} \right\rangle + |\Gamma F(r, S_l)|.$$

An algorithm for calculating $|\Gamma F(r, S_l)|$ is easily expressed in terms of the generalized binomial coefficients. One first gets the *l-canonical representation of r* by taking $r(l)$ to be the largest integer such that $r \geq \langle {r(l) \atop l} \rangle$, then taking $r(l-1)$ to be the largest integer such that

$$r \geq \left\langle {r(l) \atop l} \right\rangle + \left\langle {r(l-1) \atop l-1} \right\rangle \quad \text{etc.},$$

until equality is attained. If

$$r = \left\langle {r(l) \atop l} \right\rangle + \cdots + \left\langle {r(t) \atop t} \right\rangle,$$

t will be ≥ 1 and

$$|\Gamma CH_l| = \left\langle {r(l) \atop l-1} \right\rangle + \cdots + \left\langle {r(t) \atop t-1} \right\rangle.$$

This is just the multiset version [11, p. 170; 7] of the Kruskal–Katona theorem [14, 12]. We will use $\Gamma_l m$ to denote $|\Gamma CH_l|$, where $|CH_l| = m$. It of course depends on $k_1, \ldots, k_n$.

For example, with $k = 3$, $S = S(2, 1, 1)$, $l = 2$, and $m = 11$, we have $11 = 2\langle {3 \atop 2} \rangle + 3$ (see Fig. 2). The 2-canonical representation of 3 is $3 = \langle {2 \atop 2} \rangle + \langle {1 \atop 1} \rangle$, so

$$\Gamma_2 11 = 2\left\langle {3 \atop 1} \right\rangle + \left\langle {2 \atop 1} \right\rangle + \left\langle {1 \atop 0} \right\rangle = 6 + 2 + 1,$$

as can be verified directly from Fig. 3.

3. Applications

If H is a k-family in $S(k)$ (or more properly, H is a k-family of subsets of $M(k)$), the numbers $|H_0|, |H_1|, \ldots, |H_{K_n}|$ are called its *rank parameters*. We will henceforth write K in place of $K_n = k_1 + \cdots + k_n$.

Theorem 2. *Non-negative integers $P_0, P_1, \ldots, P_K$ are the rank parameters of a k-family in $S(k)$ if and only if*

$$P_i \leq \left\langle {n \atop i} \right\rangle \quad \textit{and} \quad N_i \leq k\left\langle {n \atop i} \right\rangle, \quad i = 1, \ldots, K,$$

where

$$N_K = P_K,$$

and

$$N_{K-j} = \Gamma_{K-j+1} N_{K-j+1} + P_{K-j}, \quad j = 1, 2, \ldots, K. \tag{1}$$

The $k=1$ case of this theorem was given by Clements [2]. The special case corresponding to ordinary sets was also given independently by Daykin, Godfrey and Hilton [10]. Then Daykin characterized the rank parameters of k-families of ordinary sets [9, Theorem 12] and Clements verified that his arguments work as well for multisets [5, Theorem 12]. Theorem 2 is a reformulation of that result in terms of Γ (more precisely, $k\Gamma$). The following example illustrates the sufficiency of the condition (1); the proof of its necessity requires arguments of Daykin.

Example 1. It follows from Theorem 2 that the numbers 1, 1, 3, 2, 0 are the rank parameters of a 3-family in $S(2, 1, 1)$ since (see Fig. 3)

$$\begin{aligned}
N_4 &= P_4 && = 0 && = 0 && \leqslant 3 = 3\left\langle {3 \atop 4} \right\rangle,\\
N_3 &= \Gamma_4 0 + P_3 && = |\Gamma F(0, 3S_4)| + 2 && = 0 + 2 = 2 && \leqslant 9 = 3\left\langle {3 \atop 3} \right\rangle,\\
N_2 &= \Gamma_3 2 + P_2 && = |\Gamma F(2, 3S_3)| + 3 && = 3 + 3 = 6 && \leqslant 12 = 3\left\langle {3 \atop 2} \right\rangle,\\
N_1 &= \Gamma_2 6 + P_1 && = |\Gamma F(6, 3S_2)| + 1 && = 5 + 1 = 6 && \leqslant 9 = 3\left\langle {3 \atop 1} \right\rangle,\\
N_0 &= \Gamma_1 6 + P_0 && = |\Gamma F(6, 3S_1)| + 1 && = 2 + 1 = 3 && \leqslant 3 = 3\left\langle {3 \atop 0} \right\rangle.
\end{aligned}$$

To obtain an example of such a 1-family, take the first $P_4 = 0$ elements of $3S_4$, then the first $P_3 = 2$ elements of $3S_3 \backslash \Gamma F(N_4, 3S_4)$, then the first $P_2 = 3$ elements of $3S_2 \backslash \Gamma F(N_3, 3S_3)$, etc. to obtain the set

$$\{(210;1), (201;1), (011;1), (200;2), (110;2), (001;2), (000;3)\},$$

where $(x; i)$ denotes the element x in kS^i. This is a 1-family in $3S$, and since $P_i \leqslant \langle {n \atop i} \rangle$ for $i = 0, 1, \ldots, K$, the x parts of our selections $(x; i)$ are distinct. If the copy indices are now dropped, the resulting set is a 3-family in S with the given rank parameters. It may be regarded as the union of the 3 disjoint 1-families $\{210, 201, 011\}$, $\{200, 110, 001\}$ and $\{000\}$. Thus a k-family induces k 1-families. Daykin has pointed out this situation for ordinary sets [9, p. 87]. Actually, a converse holds, as we now see.

Theorem 3. *There exists a k-family in $S(k)$ with rank parameters $P_0, P_1, \ldots, P_K$ if and only if*

$$0 \leqslant P_i \leqslant \left\langle {n \atop i} \right\rangle, \quad i = 0, 1, \ldots, K,$$

and there exist k 1-families in S (some of which may be empty) with rank parameters $p_0^j, p_1^j, \ldots, p_K^j$, $j = 1, 2, \ldots, k$, such that $\sum_{j=1}^k p_l^j = P_l$ for $l = 0, 1, 2, \ldots, K$. (2)

Proof. To establish the necessity of (2), suppose that we have a k-family with parameters $P_0, P_1, \ldots, P_K$. By Theorem 2, these parameters satisfy (1). We construct the required k 1-families in the manner illustrated in Example 1.

We begin by constructing a 1-family H in kS with parameters $P_0, P_1, \ldots, P_K$. At the ith stage of our construction we take P_{K+1-i} consecutive elements of kS_{K+1-i}, $i = 1, 2, \ldots, K+1$. The fact that $P_{K+1-i} \leqslant \langle {n \atop K+1-i} \rangle$ ensures that the x parts of our choices $(x; i)$ are distinct. We first choose H_K, the P_K elements of rank K. An element $(x; i) \in kS^i$ is in the *ideal* $I(G)$ of a subset G of kS if $(x; i)$ is contained in an element of $G \cap kS^i$; $(x; i) \in kS^i$ is *contained* in $(y; i) \in kS^i$ if $x_i \leqslant y_i$, $i = 1, \ldots, n$. Since we are constructing a 1-family in kS, when selecting H_l for $l < K$, we will have to avoid $I(H_K) \cap kS_l$, so it behooves us to keep $I(H_K) \cap kS_l$ as small as possible. In view of Theorem 1, this is accomplished by taking H_K to be the first P_K elements in kS_K, or, since $N_K = P_K$, $H_K = F(N_K, kS_K)$. The P_{K-1} elements H_{K-1} must now be selected from $kS_{K-1} \setminus \Gamma H_K$. To keep $(I(H_K \cup H_{K-1})) \cap kS_l$ as small as possible for $0 \leqslant l \leqslant K-2$, we take $H_{K-1} = F(N_{K-1}, kS_{K-1}) \setminus \Gamma F(N_K, kS_K)$, where $N_{K-1} = \Gamma_K N_K + P_{K-1}$. This selection is possible since $N_{K-1} \leqslant k \langle {n \atop K-1} \rangle = |kS_{K-1}|$ by (1).

Continuing in this way, taking $H_{K-j} = F(N_{K-j}, kS_{K-j}) \setminus \Gamma F(N_{K-j+1}, kS_{K-j+1})$, where $N_{K-j} = \Gamma_{K-j+1} N_{K-j+1} + P_{K-j}$, $j = 1, 2, \ldots, K$, we obtain a 1-family H in kS with the required parameters.

Since H is a 1-family in kS, if $H \cap kS^i = \{(x_1; i), (x_2; i), \ldots, (x_u; i)\}$, then $C^i = \{x_1, x_2, \ldots, x_u\}$ is a 1-family in S, $i = 1, 2, \ldots, k$. Also, if the rank parameters of C^i are $p_0^i, p_1^i, \ldots, p_K^i$, $i = 1, 2, \ldots, k$, we have $\sum_{i=1}^k p_l^i = P_l$ for $l = 0, 1, \ldots, K$. We thus have the required k 1-families. (Also, it follows from $P_l \leqslant \langle {n \atop l} \rangle$, $l = 0, 1, \ldots, K$, that these 1-families are actually mutually disjoint.)

Conversely, assuming (2) (where the k 1-families are not necessarily disjoint), we show that $P_0, P_1, \ldots, P_K$ are the rank parameters of a k-family in S. Let H^i be a 1-family with rank parameters $p_0^i, p_1^i, \ldots, p_K^i$ in kS^i, the ith copy of S, $i = 1, 2, \ldots, k$. Then $H = \bigcup_{i=1}^k H^i$ is a 1-family in kS with parameters $P_0, P_1, \ldots, P_K$, and therefore for $j = 0, 1, \ldots, K$,

$$H_{K-j} \cap G_{K-j} = \emptyset, \tag{3}$$

where $G_K = \emptyset$ and $G_{K-j} = I(H_{K-j+1} \cup H_{K-j+2} \cup \cdots \cup H_K) \cap kS_{K-j}$ for $j = 1, 2, \ldots, K$. We now construct the *compression* C of H inductively. Take C_K to be the 1-family in kS consisting of the first P_K elements of kS_K. This is possible because $P_K \leqslant \langle {n \atop K} \rangle$ by (2). Let $D_K = \emptyset$ and $D_{K-1} = I(C_K) \cap kS_{K-1}$. This is an initial segment of kS_{K-1} because C_K is an initial segment of kS_K and $D_{K-1} = \Gamma C_K$. By Theorem 1,

$$|D_{K-1}| = |\Gamma C_K| \leqslant |\Gamma H_K| = |I(H_K) \cap kS_{K-1}| = |G_{K-1}|.$$

Now for fixed l $1 \leq l \leq K$, suppose that $C_{K-j} \subset kS_{K-j}$, $j=0, 1, \ldots, l-1$ have been selected so that $C_{K-l+1} \cup C_{K-l+2} \cup \cdots \cup C_K$ is a 1-family in kS,

$$D_{K-l} = I(C_{K-l+1} \cup C_{K-l+2} \cup \cdots \cup C_K) \cap kS_{K-l}$$

is an initial segment of kS_{K-l} and $|D_{K-l}| \leq |G_{K-l}|$. In view of (3), $H_{K-l} \subset kS_{K-l} \setminus G_{K-l}$ and therefore

$$\begin{aligned} P_{K-l} = |H_{K-l}| &\leq k\left\langle {n \atop K-l} \right\rangle - |G_{K-l}| \\ &\leq k\left\langle {n \atop K-l} \right\rangle - |D_{K-l}| = |kS_{K-l} \setminus D_{K-l}|. \end{aligned} \tag{4}$$

We can therefore define C_{K-l} to be the first P_{K-l} elements of $kS_{K-l} \setminus D_{K-l}$ and have $C_{K-l} \cup C_{K-l+1} \cup \cdots \cup C_K$ be a 1-family in kS. Also, $D_{K-l} \cup C_{K-l}$ is an initial segment of kS_{K-l} so, if $l < K$,

$$\begin{aligned} \Gamma(D_{K-l} \cup C_{K-l}) &= \Gamma(I(C_{K-l+1} \cup \cdots \cup C_K) \cap kS_{K-l} \cup C_{K-l}) \\ &= I(C_{K-l} \cup C_{K-l+1} \cup \cdots \cup C_K) \cap kS_{K-l-1} \\ &= D_{K-l-1} \end{aligned}$$

is an initial segment of kS_{K-l-1}. In terms of cardinalities, this reads

$$\Gamma_{K-l} |D_{K-l} \cup C_{K-l}| = |D_{K-l-1}|, \quad 0 \leq l < K. \tag{5}$$

By (5), (3) and Theorem 1,

$$\begin{aligned} |D_{K-l-1}| &= \Gamma_{K-l}(|D_{K-l}| + P_{K-l}) \\ &\leq \Gamma_{K-l}(|G_{K-l}| + P_{K-l}) = \Gamma_{K-l} |G_{K-l} \cup H_{K-l}| \\ &\leq |\Gamma(G_{K-l} \cup H_{K-l})| = |G_{K-l-1}|. \end{aligned}$$

We are thus able to define C_i for $i=K, K-1, \ldots, 0$ so that their union is a 1-family in kS.

We now verify that the rank parameters $P_0, P_1, \ldots, P_K$ of C satisfy (1) and are therefore (Theorem 2) the parameters of a k-family in S.

By hypothesis, $P_l \leq \langle {n \atop l} \rangle$, $l=0, 1, \ldots, K$. The numbers N_{K-j} involved in (1) satisfy

$$N_{K-j} = |D_{K-j} \cup C_{K-j}|, \quad j=0, 1, \ldots, K,$$

since $N_K = P_K = |D_K \cup C_K|$ if $j=0$, and if the formula holds for $j=0, 1, \ldots, l-1 < K$, it follows from (5) that

$$\begin{aligned} N_{K-l} &= \Gamma_{K-l+1} N_{K-l+1} + P_{K-l} = \Gamma_{K-l+1} |D_{K-l+1} \cup C_{K-l+1}| + P_{K-l} \\ &= |D_{K-l}| + P_{K-l} = |D_{K-l} \cup C_{K-l}|. \end{aligned}$$

Then for $0 \leq l \leq K$ we have

$$
\begin{aligned}
N_{K-l} &= \Gamma_{K-l+1} N_{K-l+1} + P_{K-l} \\
&= \Gamma_{K-l+1} |D_{K-l+1} \cup C_{K-l+1}| + P_{K-l} \\
&= |D_{K-l}| + P_{K-l} \leqslant k\left\langle {n \atop K-l} \right\rangle,
\end{aligned}
$$

where the inequality follows from (4). Thus (1) holds and Theorem 3 follows. □

Example 2. By Theorem 2, $0,1,0,1,0$ $0,0,2,1,0$, and $0,2,1,0,0$ are the parameters of 1-families in $S(2, 1, 1)$ (see Fig. 1). Then by Theorem 3, $0,3,3,2,0$ are the parameters of a 3-family in $S(2, 1, 1)$. The canonical example, written as a union of 3 disjoint 1-families is $\{210, 201, 011\} \cup \{200, 110, 001\} \cup \{100, 010\}$ (see Fig. 3).

The case of the following corollary of Theorem 3 corresponding to ordinary sets ($k_1 = 1$) and $k = 1$ was conjectured by Kleitman and Milner [12, p. 147], proved by Daykin, Godfrey and Hilton [10] and extended to multisets by Clements [4]. Daykin [10, p. 94] conjectured the case corresponding to ordinary sets and arbitrary k.

Corollary. *If $P_0, P_1, \ldots, P_K$ are the rank parameters of a k-family in S and $P_l + P_{K-l} \leqslant \langle {n \atop l} \rangle$ for $0 \leqslant l \leqslant K$ and $l \neq \frac{1}{2}K$, then*

$$
P'_l = \begin{cases} 0, & \text{for } 0 \leqslant l < \frac{1}{2}K \\ P_l + P_{K-l}, & \text{for } \frac{1}{2}K < l \leqslant K, \\ P_{\frac{1}{2}K}, & \text{for } l = \frac{1}{2}K \text{ (if } K \text{ is even)} \end{cases}
$$

are the rank parameters of a k-family in S.

Proof. By Theorem 3, there exist k 1-families in S with parameters $p_0^j, p_1^j, \ldots, p_K^j$, $j = 1, 2, \ldots, k$, such that $\sum_{j=1}^{k} p_l^j = P_l$, $l = 0, 1, \ldots, K$. Then for $i \leqslant j \leqslant k$ it is known [4] that

$$
q_l^j = \begin{cases} 0, & \text{for } 0 \leqslant l < \frac{1}{2}K, \\ p_l^j + p_{K-l}^j, & \text{for } \frac{1}{2}K < l \leqslant K, \\ p_{\frac{1}{2}K}^j, & \text{for } l = \frac{1}{2}K \text{ (if } K \text{ is even)} \end{cases}
$$

are parameters of a 1-family in S. Hence, (Theorem 3)

$$
P'_l = \sum_{j=1}^{k} q_l^j \leqslant \left\langle {n \atop l} \right\rangle, \quad l = 0, 1, \ldots, K
$$

are parameters of a k-family in S. □

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